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Matrix Fraction Approach for Finite-State Aerodynamic Modeling

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A least-square procedure for the finite-state approximation of the aerodynamic matrix is introduced, which is more efficient and/or simpler to use than those currently available. This is accomplished by starting from the approximation of the aerodynamic matrix as ND^{-1} or $D^{-1}N$, where N and D are matrices with polynomial dependence on complex reduced frequency p. Three different finite-state realizations based on the same matrix polynomial approximation are presented. The advantages of the approach and problems encountered in using the method are discussed. Numerical results are included.

Nomenclature

A	= state-space coefficient matrix
B, b	= minimum problem coefficient matrix, Eq. (26)
$D, D_i (i = 0, 1, \ldots, M-1)$	= matrices defined in Eqs. (6) and (7)
$\bar{D}$	= approximate aerodynamic matrices in Karpel modeling as defined by Eq. (5)
$E_i (i = 0, 1, 2)$	= approximate aerodynamic matrices as defined by Eqs. (4), (5), (9), (15), (17), and (18)
$E(p)$	= aerodynamic matrix
$\hat{E}(p)$	= approximated aerodynamic matrix
$F, F_i (i = 1, 2, \ldots, M)$	= approximate aerodynamic matrices as defined by Eqs. (9) and (15)
$\hat{F}_i (i = 1, 2, \ldots, M)$	= approximate aerodynamic matrices in Roger modeling as defined by Eqs. (4)
$\bar{F}$	= approximate aerodynamic matrices in Karpel modeling as defined by Eq. (5)
f	= generalized aerodynamic force vector
$G, G_i (i = 1, 2, \ldots, M)$	= approximate aerodynamic matrices as defined by Eqs. (9) and (15)
$\bar{G}$	= approximate aerodynamic matrices in Karpel modeling as defined by Eq. (5)
I	= identity matrix
k	= reduced frequency, $\mathcal{I}m(p) = \omega\ell/U_\infty$
ℓ	= reference length
M_∞	= Mach number
$N, N_i (i = 0, 1, \ldots, M+2)$	= matrices defined in Eqs. (6) and (7)
p	= complex reduced frequency, $s\ell/U_\infty$
q	= Lagrangian variable vector
q_D	= dynamic pressure
r, r_i	= aerodynamic state space vectors
$\mathcal{S}$	= body surface
s	= Laplace variable
t	= aerodynamic force per unit area
U_∞	= flight speed
v, v_i	= aerodynamic state space vectors
W	= diagonal matrix of the weights, Eq. (26)
$w(k)$	= weight function Eq. (11)
x	= space-state vector
$Z(p)$	= matrix defined in Eq. (12)
$\beta_k, (k = 1, 2, \ldots, M)$	= Roger approximation coefficients Eq. (4)
ε	= error function defined in Eq. (11)
ϱ_∞	= density of the undisturbed air
Φ_n	= nth mode shape function
ω	= frequency, $\mathcal{I}m(s)$
Ω	= natural frequency diagonal matrix
AR	= aspect ratio
tr[ ]	= trace of a matrix
$[\check{\ }]$	= see Eq. (30)
[ ]*	= Hermitian adjunct of a matrix

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Introduction

FINITE-state aerodynamic modeling denotes a process whereby the aerodynamic transfer function is approximated by means of appropriate rational expressions. Probably, the earliest example of this approach is the work of Jones[1] who gives a rational approximation for the Theodorsen function and the corresponding time-domain approximation for the Wagner function. More recent examples of this approach include Refs. 2–11, where closely related techniques are used to obtain approximate expressions either of the Theodorsen function or directly of the aerodynamic matrix. As a result, the aeroelastic model [see Eq. (1)] may be rewritten, in the time domain, as $\dot{x} = Ax$, with apparent advantages for the aeroelastic analysis (for instance, the V-g method may be replaced by a simple root locus) and for feedback-control utilization (for instance, in the use of optimal control techniques). Of particular interest here are the works of Roger[3] and Karpel,[6] which are examined more closely later in this section, after the introduction of some basic definitions.

Consider an aeroelastic system described in terms of the amplitudes $q_n(t)$ of the natural modes of vibration $\boldsymbol{\Phi}_n$ which are here assumed to be normalized so as to have the generalized masses equal one. The corresponding Lagrange equations of motion, neglecting structural damping and gravity, are given by[12]

$$\frac{\mathrm{d}^2\boldsymbol{q}}{\mathrm{d}t^2} + \boldsymbol{\Omega}^2\boldsymbol{q} = q_D\boldsymbol{f} \tag{1}$$

where $\boldsymbol{\Omega}$ is the diagonal matrix of the natural frequencies of vibration of the structure, $q_D = \varrho_\infty U_\infty^2/2$ is the dynamic pressure, whereas the components of $\boldsymbol{f}$ are the generalized aerodynamic forces associated with the nth mode, $\boldsymbol{\Phi}_n (n = 1, \ldots, N)$, as

$$q_D f_n = \oiint_S \boldsymbol{t} \cdot \boldsymbol{\Phi}_n \, \mathrm{d}S \tag{2}$$

where $\boldsymbol{t}$ is the aerodynamic force per unit area acting on S.

Here, we assume that the aerodynamic forces depend linearly upon the Lagrangian coordinates $q_n(t)$; specifically, in the following we limit ourselves to potential subsonic or supersonic flows (the modeling of viscous effects, particularly important for the control surfaces, and/or of transonic effects, falls beyond the scope of the present paper). Hence, the Laplace transform of the generalized force vector can be expressed as (in the following the Laplace transform of a time-dependent function $f(t)$ will be indicated as $\tilde{f}(s)$, where s is the Laplace variable)

$$\tilde{\boldsymbol{f}}(s) = \boldsymbol{E}(s\ell/U_\infty)\tilde{\boldsymbol{q}}(s) \tag{3}$$

where $\boldsymbol{E}$ is the so-called aerodynamic matrix. As is well known and as emphasized in Eq. (3), $\boldsymbol{E}$ is a function of s and U_∞ only through the variable $p := s\ell/U_\infty$, which is known as the complex reduced frequency. Note that $\boldsymbol{E}$ may be obtained analytically for some simple cases (e.g., classic Theodorsen incompressible two-dimensional aerodynamic theory); otherwise, $\boldsymbol{E}(p)$ is evaluated numerically, for instance, by lifting-surface, doublet-lattice, or panel methods (see, e.g., Ref. 13). More precisely, the algorithm for the evaluation of $\boldsymbol{E}(p)$ is typically available only along the imaginary axis; $\boldsymbol{E}(p)$ is then the analytic continuation of $\boldsymbol{E}(ik)$ (see comment 4 in the Appendix).

Next, consider the work of Roger[3] and of Karpel.[6] In the finite-state aerodynamic model proposed by Roger,[3] the aerodynamic matrix $\boldsymbol{E}(p)$ [see Eq. (3)] is approximated as

$$\boldsymbol{E}(p) \simeq \hat{\boldsymbol{E}}(p) := \boldsymbol{E}_2 p^2 + \boldsymbol{E}_1 p + \boldsymbol{E}_0 + \sum_{k=1}^{M} \frac{p}{p+\beta_k}\hat{\boldsymbol{F}}_k \tag{4}$$

where the poles β_k are real and positive (to ensure stability) and are prescribed as functions of the range of interest of p, whereas the matrices are evaluated by using a least-square technique on the individual terms of the matrix $\boldsymbol{E}$. The resulting aeroelastic model contains NM additional state variables for the aerodynamics, where N denotes the number of modes. Further developments as well as applications of this model, for instance, to flutter suppression, are given in Refs. 14–19.

It is apparent that it is highly desirable to improve the accuracy of the approximation without increasing the number of the state variables. This may be accomplished by increasing the dimensions of the minimization space, although keeping the number of variables constant. An example of this approach is the model introduced by Karpel[6] who proposes, for the aerodynamic matrix, the structure

$$\boldsymbol{E}(p) \simeq \hat{\boldsymbol{E}}(p) := \boldsymbol{E}_2 p^2 + \boldsymbol{E}_1 p + \boldsymbol{E}_0 + \bar{\boldsymbol{D}}(p\boldsymbol{I} + \bar{\boldsymbol{G}})^{-1}\bar{\boldsymbol{F}} p \tag{5}$$

where $\bar{\boldsymbol{D}}$ is an $N \times N_1$ matrix, $\bar{\boldsymbol{F}}$ is an $N_1 \times N$ matrix, and $\bar{\boldsymbol{G}}$ is an $N_1 \times N_1$ diagonal matrix, where N_1 is arbitrary; the elements of $\bar{\boldsymbol{G}}$ are enforced to be negative to ensure stability [for the sake of consistency with the notation used in the remainder of the paper, the following changes in the notation of Karpel[6] have been adopted: $\boldsymbol{A}_i \to \boldsymbol{E}_i (i = 0, 1, 2), \boldsymbol{D} \to \bar{\boldsymbol{D}}, \boldsymbol{R} \to -\bar{\boldsymbol{G}}$, and $\boldsymbol{E} \to \bar{\boldsymbol{F}}$]. Additional developments and applications of this approach are given in Refs. 20–25.

As already noted by Karpel,[25] the Roger's model is a particular case of Karpel's. Indeed, denoting $\boldsymbol{I}_N$ the unit matrix of dimensions $N \times N$, we have, choosing $\bar{\boldsymbol{F}} = [\hat{\boldsymbol{F}}_1|\hat{\boldsymbol{F}}_2|\ldots|\hat{\boldsymbol{F}}_M]$, and $\bar{\boldsymbol{D}} = [\boldsymbol{I}_N|\boldsymbol{I}_N|\ldots|\boldsymbol{I}_N]$, and $\bar{\boldsymbol{G}}(NM \times NM)$ to be a diagonal matrix with M diagonal blocks of type $-\beta_j\boldsymbol{I}_N$, that Eq. (5) reduces to Eq. (4). Note that Karpel's model requires additional N_1 state variables. If $N_1 = N$, the number of state variables is equal to those of Roger[3] with $M = 1$; however, in this case, the number of degrees of freedom in the least-square process of Karpel[6] exceeds those of Roger[3] by N^2 (see also the Appendix, comment 8). For this reason, the model of Karpel[6] is expected to be more accurate than that of Roger.[3] The only disadvantage of Karpel's model is the complexity of the solution of the least-square problem, which requires the use of an iterative technique to solve a system of nonlinear algebraic equations.

The complexity of the procedure used by Karpel to obtain the finite-state formulation may be reduced by approximating the matrix $\boldsymbol{E}$ as

$$\boldsymbol{E}(p) \simeq \hat{\boldsymbol{E}}(p) = \boldsymbol{N}(p)\boldsymbol{D}^{-1}(p) := \left(\sum_{i=0}^{M+2}\boldsymbol{N}_i p^i\right)\left(\sum_{i=0}^{M}\boldsymbol{D}_i p^i\right)^{-1} \tag{6}$$

(with $\boldsymbol{D}_M = \boldsymbol{I}_N$) or as

$$\boldsymbol{E}(p) \simeq \hat{\boldsymbol{E}}(p) = \boldsymbol{D}^{-1}(p)\boldsymbol{N}(p) := \left(\sum_{i=0}^{M}\boldsymbol{D}_i p^i\right)^{-1}\left(\sum_{i=0}^{M+2}\boldsymbol{N}_i p^i\right) \tag{7}$$

(with $\boldsymbol{D}_M = \boldsymbol{I}_N$) and by imposing that

$$\boldsymbol{Z} := \boldsymbol{E}\boldsymbol{D} - \boldsymbol{N} \quad \text{or} \quad \boldsymbol{Z} := \boldsymbol{D}\boldsymbol{E} - \boldsymbol{N} \tag{8}$$

respectively, be small in some least-square sense. The minimization problem then yields a system of linear algebraic equation in the minimization parameters [see Eq. (11) and discussion that follows, as well as the Appendix, in particular, comment 1, where the optimization technique is discussed, and comment 2, where the fact that the leading term is p^2 is also examined; however, see also comment 6].

This approach was proposed independently (and, incidentally, at the same conference), by Morino et al.[9] and by Ghiringhelli and Mantegazza.[10] Whereas the basic formulations coincide, the implementations, in particular the higher order approach, differ considerably. The aim of the present paper is to present in a unified fashion the concepts introduced in Refs. 9 and 10, some more recent developments, as well as clarifications which resulted from the comparison of the two approaches. The expressions used in Eqs. (6) and (7) are known as matrix-fraction representation[26]; hence the name matrix fraction approach used here to indicate the specific method proposed in this paper for approximating the aerodynamic matrix to yield a finite-state aerodynamic model. It may be noted that Vepa[2] notes the possibility of using Eq. (6); however, he does not connect this with the optimization technique, nor does he discuss how to connect this with a state-variable approach.

As apparent from Ref. 9, the first three authors are responsible for the content of the section entitled Basic Model, and of the first portion of the sections entitled Higher Order models [Eqs. (15) and (16)] and Numerical Results (Figs. 1–4); specifically, Morino for suggesting the use of Eq. (9), De Troia for introducing the use of Eq. (14) and Mastroddi for the extension to the general case [Eqs. (15) and (16); not included in this paper are certain applications of the formulation to aeroservoelastic problems, which involved a collaboration with Modesto Pecora of ALENIA, for which the reader is referred to Ref. 9. On the other hand, as apparent from Ref. 10, Ghiringhelli and Mantegazza are responsible for the remainders of the sections entitled Higher Order models [Eqs. (17–23) and Numerical Results (Figs. 5 and 6). The Appendix was developed jointly by all of the authors.

Basic Model

The basic model proposed in Ref. 9 consists of expressing the aerodynamic matrix $E(p)$ as

$$E(p) \simeq \hat{E}(p) := p^2 E_2 + pE_1 + E_0 + (pI + G)^{-1} F p \qquad (9)$$

where G and F are full square matrices which are independent of p [which may be recast as a particular case of Eq. (7) with $M = 1$].

The aeroelastic system resulting from Eqs. (1), (3), and (9) is given by, in the time domain (see also the Appendix, comment 3),

$$\begin{gathered}\left(U_\infty^2/\ell^2\right)\ddot{q} + \Omega^2 q = q_D(E_2\ddot{q} + E_1\dot{q} + E_0 q + r)\\ \dot{r} + Gr = F\dot{q}\end{gathered} \qquad (10)$$

with $r(0) = 0$ and where the overdot denotes differentiation with respect to $U_\infty t/\ell$. As shown in the Appendix, comment 8, the representation given by Eq. (9) is fully equivalent to that of Karpel[6] for the case in which $N = N_1$. The advantage of this form with respect to that of Karpel[6] is the simplicity with which the matrices E_h, G, and F are obtained by a least-square approach. This consists of setting (see the Appendix, comment 2, for details)

$$\varepsilon^2 = \int_0^{+\infty} w(k)\mathrm{tr}[Z^*(ik)Z(ik)]\,\mathrm{d}k = \min \qquad (11)$$

where $k = \Im m(p)$ and

$$Z(p) := (pI + G)\left[p^2E_2 + pE_1 + E_0 - E(p)\right] + pF \qquad (12)$$

whereas $w(k)$ is a suitable weight function in the frequency range of interest. In addition, tr denotes the trace of the matrix, and Z^* denotes the Hermitian adjunct of Z (complex conjugate of the transpose of Z). Note that from Eq. (9), we have

$$\hat{E}(p)|_{p=0} = E_0 \qquad (13)$$

Accordingly, in the following E_0 is considered as prescribed from Eq. (13) (and not available for the minimization process). Thus, Eq. (12) may be rewritten as

$$Z(p) = p^3N_3 + p^2N_2 + pN_1 + GE_0 - (pI + G)E(p) \qquad (14)$$

where $N_3 = E_2, N_2 = E_1 + GE_2$, and $N_1 = E_0 + GE_1 + F$. Equation (14) is a particular case of the second of Eqs. (8) with $M = 1, GE_0 = N_0$ and $G = D_0$.

The least-square minimization in Eq. (11) is performed with respect to G, N_1, N_2, and N_3. Note that the error function in Eq. (11) is a quadratic function of the elements of these matrices (see, again, the Appendix, comment 1). Therefore, the least-square process yields a system of linear algebraic equation. Once G and $N_j (j = 1, 2, 3)$ have been obtained, the preceding expressions for N_3, N_2, N_1 may be used sequentially to yield E_2, E_1, and F, respectively.

Higher Order Models

Next, consider the higher order extension, which is based on Eqs. (6) and (7). The approach introduced in Ref. 9 is to arrive at Eq. (7) by starting from the approximation

$$E(p) \simeq \hat{E}(p) = p^2E_2 + pE_1 + E_0 + (pI + G_1)^{-1}\left[pF_1 + (pI + G_2)^{-1}\left[pF_2 + \cdots\left[pF_{M-1} + (pI + G_M)^{-1}pF_M\right]\ldots\right]\right] \qquad (15)$$

where $\mathbf{E}_i (i = 0, 1, 2), \mathbf{F}_j$, and $G_j (j = 1, 2, \ldots M)$ are again full matrices independent of p. The corresponding aeroelastic system may be written as (the overdot retains its earlier definition)

$$\begin{gathered}\frac{U_\infty^2}{\ell^2}\ddot{q} + \Omega^2 q = q_D(E_2\ddot{q} + E_1\dot{q} + E_0 q + r_1)\\ \dot{r}_1 + G_1 r_1 = F_1\dot{q} + r_2\\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{r}_{M-1} + G_{M-1}r_{M-1} = F_{M-1}\dot{q} + r_M\\ \dot{r}_M + G_M r_M = F_M\dot{q}\end{gathered} \qquad (16)$$

The relationships between the matrices $D_i (i = 0, 1, \ldots, M-1)$ and $N_j (j = 0, 1, \ldots, M+2)$ and the matrices E_i, F_j, and G_j are given in Ref. 9; in particular, for the case $M = 2$, we have $N_4 = E_2, N_3 = E_1 + (G_1 + G_2)E_2, N_2 = E_0 + (G_1 + G_2)E_1 + G_2G_1E_2 + F_1, N_1 = (G_1 + G_2)E_0 + G_2G_1E_1 + F_2 + G_2F_1, N_0 = D_0E_0, D_1 = G_1 + G_2$, and $D_0 = G_2G_1$. In this case, the minimization problem for evaluating the matrices $N_j (j = 1, \ldots, 4)$ and $D_i (i = 0, 1)$ is similar to that described in the preceding section and in Appendix, comment 1; the equations resulting from the minimization problem are linear in the elements of N_j and D_j. Once these matrices have been evaluated, then G_1 is obtained (see the earlier expressions for D_1 and D_0) from the equation $G_1^2 - D_1G_1 + D_0 = 0$ (in the numerical applications this equation is solved numerically with a Newton–Raphson method). Then, the given expression for D_1 yields G_2. Finally, the expressions for N_4, N_3, N_2, N_1, and N_0 in the preceding may be used sequentially to yield E_2, E_1, F_1, and F_2, respectively. It should be noted that the matrix G_1 is not unique. Correspondingly, G_2 and F_2 are not unique (whereas E_1, E_2, and F_1 are unique). However, the corresponding systems are equivalent because they all correspond to the same matrix $\hat{E}(p) = D^{-1}N$ [Eq. (7)] and the solution for the matrices $N_j (j = 1, \ldots, 4)$ and $D_j (j = 1, 2)$ is unique, as it is obtained by a linear procedure.

For $M > 2$, the relationships between the matrices $D_i (i = 0, 1, \ldots, M-1)$ and $N_j (j = 0, 1, \ldots, M+2)$ and the matrices E_i, F_j, and G_j become quite cumbersomes. The approach suggested in Ref. 10, clearly much simpler to use, is outlined in the following. Through a polynomial division, Eqs. (6) and (7) may be rewritten as

$$\hat{E}(p) = p^2E_2 + pE_1 + E_0 + \left(\sum_{i=0}^{M-1} R_i p^i\right)\left(\sum_{i=0}^{M} D_i p^i\right)^{-1} \qquad (17)$$

and

$$\hat{E}(p) = p^2E_2 + pE_1 + E_0 + \left(\sum_{i=0}^{M} D_i p^i\right)^{-1}\left(\sum_{i=0}^{M-1} R_i p^i\right) \qquad (18)$$

respectively. Clearly, E_i, R_i, and D_i in Eq. (17) differ from those in Eq. (18) [as well as those in Eqs. (4), (5), and (15)]. Note that Eqs. (17) and (18) can be put in the form

$$\hat{E}(p) = p^2E_2 + pE_1 + E_0 + C(pI - A)^{-1}B \qquad (19)$$

by setting, respectively,

$$C^T = \begin{bmatrix} R_{M-1}^T \\ R_{M-2}^T \\ \cdot \\ \cdot \\ \cdot \\ R_1^T \\ R_0^T \end{bmatrix} \qquad B = \begin{bmatrix} I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -D_{M-1} & -D_{M-2} & \cdots & -D_1 & -D_0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \qquad (20)$$

and

$$C^T = \begin{bmatrix} I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} R_{M-1} \\ R_{M-2} \\ \cdot \\ \cdot \\ \cdot \\ R_1 \\ R_0 \end{bmatrix}$$

$$A = \begin{bmatrix} -D_{M-1} & I & 0 & \cdots & 0 \\ -D_{M-2} & 0 & I & \cdots & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ -D_1 & 0 & 0 & \cdots & I \\ -D_0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{21}$$

The form $C(pI-A)^{-1}B$ is classical and arises from the input-output relationship in the state-space approach, i.e., $\dot{x} = Ax + Bu, y = Cx$. Equations (20) and (21) are closely related to what Kailath[26] calls the block observer form and the block controller form (note that here the notation is slightly different because in Ref. 26 the polynomial matrices are represented as $R = \sum_i R_i p^{M-i}$).

The aeroelastic systems corresponding to the forms obtained by Eqs. (20) and (21) are given by

$$\begin{gathered} \frac{U_\infty^2}{\ell^2}\ddot{q} + \Omega^2 q = q_D\left(E_2\ddot{q} + E_1\dot{q} + E_0 q + \sum_{i=0}^{M-1} R_i v_i\right) \\ \dot{v}_0 = v_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{v}_{M-2} = v_{M-1} \\ \dot{v}_{M-1} + \sum_{i=0}^{M-1} D_i v_i = q \end{gathered} \tag{22}$$

and

$$\begin{gathered} \frac{U_\infty^2}{\ell^2}\ddot{q} + \Omega^2 q = q_D\left(E_2\ddot{q} + E_1\dot{q} + E_0 q + v_0\right) \\ \dot{v}_0 + D_{M-1}v_0 = v_1 + R_{M-1}q \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{v}_{M-2} + D_1 v_0 = v_{M-1} + R_1 q \\ \dot{v}_{M-1} + D_0 v_0 = R_0 q \end{gathered} \tag{23}$$

respectively. Note that Eq. (10) is closely related to Eq. (23) with $M = 1$.

It should be noted that the formulation corresponding to Eqs. (17–21) is used here only in connection with minimum-state reduction techniques (see comment 6 in the Appendix); in this case one encounters some of the complexities of the appoach used by Karpel[6] (such as iteration and stability); however, such a technique appears to be more straightforward and efficient than Karpel's.

Numerical Results

Some applications of the formulations just presented are presented in this section to assess the accuracy and the efficiency of the approach proposed here. The results presented in Figs. 1–4 are based on the approach of Ref. 9 [see Eqs. (9) and (15)], with the original data obtained with the unsteady subsonic boundary element method described in Ref. 27. On the other hand, Figs. 5 and 6 are based on the approach of Ref. 10 [see Eqs. (17–21), as well as comment 6 in

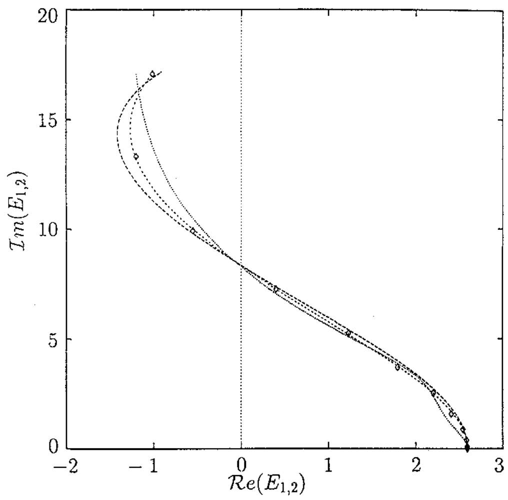

Fig. 1 $E_{1,2}$ element of aerodynamic 2 × 2 matrix (plunge and pitch mode assumed) for a rectangular wing with Æ = 2 at M_∞ = 0.5; comparison between the model with M = 1, the model with M = 2, and the Roger's method: ◇ data; ---- Eq. 15, M = 1; · · · · · Eq. 15, M = 2; …… Roger model.

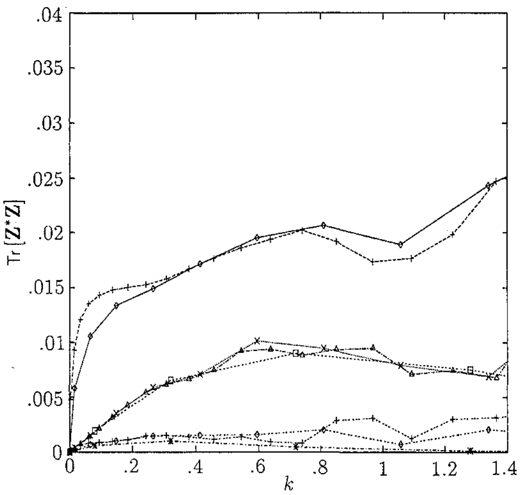

Fig. 2 Error function [Eq. (11)] relative to the higher term of the aerodynamic matrix for the case of Fig. 1 for different finite-state model, M = 1, 2, and number of sampled aerodynamic data N_S: ···□··· M = 1, N_S = 6; ···×··· M = 1, N_S = 12; ---Δ--- M = 1, N_S = 24; —◇— Roger model, N_S = 12; --+-- Roger model, N_S = 24; ---×--- M = 2, N_S = 6; ··· ◇ ··· M = 2, N_S = 12; —+— M = 2, N_S = 24.

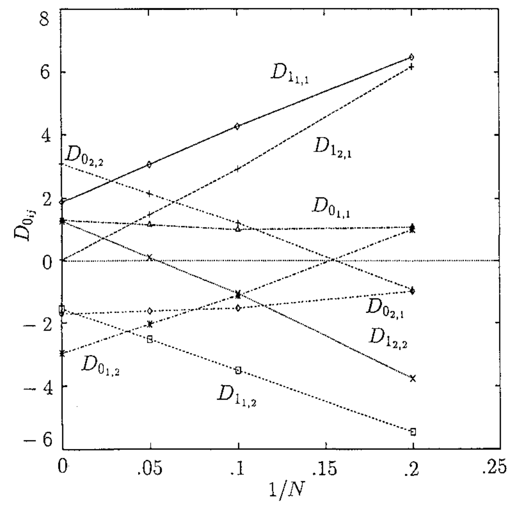

Fig. 3 Convergence analysis for the elements of the finite-state aerodynamic matrices D_1 and D_0 obtained with the finite-state modeling with M = 2, test case of Fig. 1.

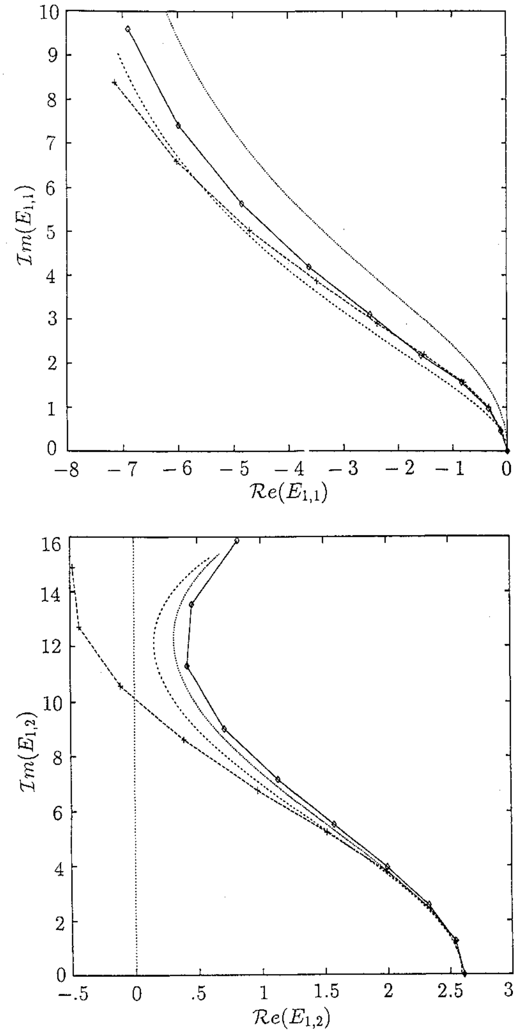

Fig. 4 **Lift coefficient due to a plunge mode $E_{1,1}$ and lift coefficient due to a pitch mode with respect to the midchord $E_{1,2}$ with the reduced frequency as implicit parameter (Δk = 0.2) for the rectangular wing of Fig. 1; comparisons of converged finite-state aerodynamic with numerical results of Refs. 28 and 29: —◇— Blair and Williams [28]; - -+- - Giesing et al. [29]; · · · · · Eq. 15, M = 1; ······ Eq. 15, M = 2.**

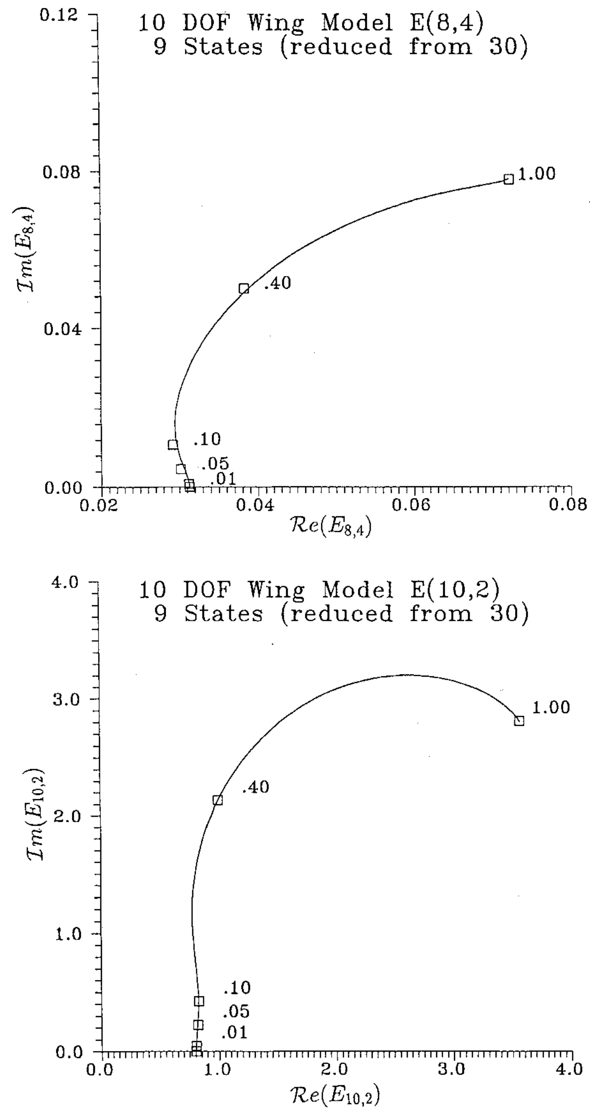

Fig. 5 **Fit of some coefficients of the aerodynamic transfer matrix for an oscillating wing, □ values from doublet lattice.**

the Appendix], with original data obtained from the doublet lattice method,[29] as available in the program NASTRAN.

Consider first Fig. 1, which presents the elements $E_{1,2}(ik)$ (lift due to a pitch mode; see Ref. 9 for the other elements of the aerodynamic matrix as well as other applications) of the aerodynamic matrix for a rectangular zero-thickness wing at $M_\infty = 0.5$, with aspect ratio $\mathcal{R} = 2$ (the modes 1 and 2 correspond to plunge and pitch around the half-chord, respectively). Specifically, Fig. 1 depicts a parametric representation of $E_{1,2}(ik)$ in the complex plane (the abscissas and the ordinates are the real and the imaginary parts of $E_{1,2}$, respectively; the parameter of each curve is the reduced frequency, $k = \omega\ell/U_\infty$ with $0 < k < 2$). Note that for this application the sum in Eq. (25) is performed with $w(k_n) = 1$ and $k_n = \kappa n^2$ with $\kappa = 0.0165$ and $n = 0, 1, \ldots, 11$ (i.e; a parabolic-stretch function around the zero; this corresponds to using a weight function $w(k) = \sqrt{\kappa/4k}$, see comment 1 in the Appendix). Two of the three curves correspond to the model of Ref. 9 with $M = 1$, and $M = 2$, respectively. The third curve corresponds to our results obtained using the model of Roger.[3] As already mentioned, the original data were obtained using the unsteady subsonic boundary element method discussed in Ref. 27.

Next, we study the effect of the number of sample points in the evaluation of the finite-state model. The error function [integrand in Eq. (11), with $w(k) = 1$ and normalized with respect to the highest modulus of the elements of the matrix **Z**] as a function of the reduced frequency k is depicted in Fig. 2 for the case of the rectangular wing of Fig. 1; the error functions corresponding to the method proposed here have lower values than those obtained with the Roger model (see Eq. (4) and Ref. 3]. Note also that all of the error functions are not as sensitive to the number of sampled data used for the analysis.

Next, we show how finite-state modeling may be used for a study of the convergence of the aerodynamic forces as the panel size goes to zero. In Fig. 3 the elements of the matrix $\boldsymbol{D}_i$ obtained with the finite-state modeling of Ref. 9 are depicted as functions of the reciprocal of the number of panels used (chordwise, spanwise, and wakewise direction; uniform paneling has been used for the mesh generation) to obtain the aerodynamic data (see Ref. 9 for the convergence analysis of the other matrices). It may be worth noting that, as pointed out in Ref. 9, for the case $M = 1$ [Eq. (9)] the converged values of the elements of the matrices $\boldsymbol{E}_2$, $\boldsymbol{E}_1$, $\boldsymbol{E}_0$, $\boldsymbol{F}$, and $\boldsymbol{G}$ may be extrapolated directly; on the other hand, for the case $M = 2$ [see Eq. (15)], this approach cannot be used because, as already mentioned, the solution for $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{F}_2$ is not unique. Nonetheless, the extrapolation may be performed on the matrices $\mathbf{N}_k$ and $\mathbf{D}_k$, since their solution is unique (linear problem). From these, the converged values for the matrices $\boldsymbol{E}_k$ ($k = 0, 1, 2$), $\boldsymbol{F}_i$ ($i = 1, 2$), $\boldsymbol{G}_j$ ($j = 1, 2$), may be evaluated as indicated earlier. A comparison between the "extrapolated" finite-state aerodynamic results ($M = 1$ and $M = 2$) and numerical results of Blair and Williams[28] and Giesing et al.[29] is shown in Fig. 4 for the elements $E_{1,1}$ and $E_{1,2}$ of the aerodynamic matrix for the same case as in Fig. 1.

Finally, we show some results obtained using the alternate finite-state aerodynamic modeling of Ref. 10 [see Eqs. (17–23) and conse-

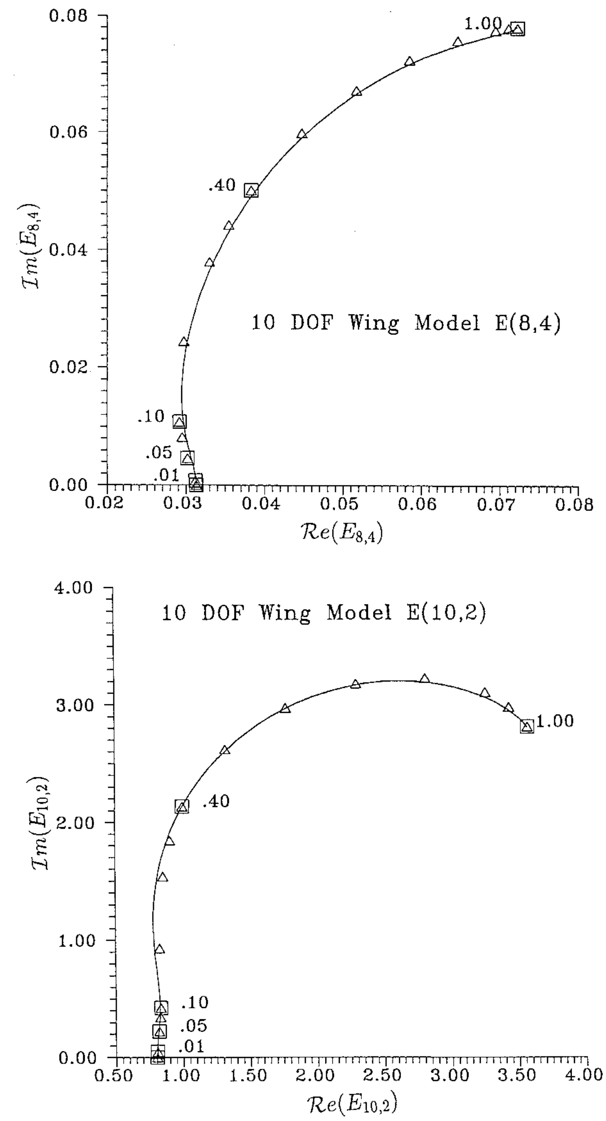

Fig. 6 Interpolation based on Eq. 28 vs minimum-state-space identification for the same example of Fig. 5; ——— state space model, □□□□ original data, ΔΔΔΔ interpolation.

quent state reduction, also see comment 6 in the Appendix]. Figure 5 demonstrates the approximation defined by 9 states, obtained after reduction of 30 initial states, for some terms of the 10×10 aerodynamic transfer matrix related to a wing to be used in an integrated optimization procedure. The results are relative to a wing having aspect ratio $\AE = 13$ and sweep angle $\Lambda = 15$ deg, which is equipped with an outboard trailing-edge aileron. As mentioned earlier, the original data are obtained using doublet lattice method,[29] as available in the program NASTRAN; the wing is modeled with 10 panels along the chord direction and 15 along the span extension whereas the aileron is modeled with 5×6 elements.[30] Finally, the use of Eq. (28) in generating new points (see comment 5 in the Appendix) is demonstrated in Fig. 6 against the minimum state space identification used for the example of Fig. 5.

Concluding Remarks

A least-square matrix-fraction approximation of the aerodynamic matrix is presented which allows one to describe the aerodynamics in a finite-state form. The low-order model presented here is simpler to use than Karpel's, as it requires only the minimization of a quadratic functional, i.e., the resolution of a linear problem. Moreover, the use of higher order formulations in connection with minimum-state reduction techniques has been examined; in this case some of the issues experienced by Karpel[6] (iteration and stability) were encountered, but the overall procedure appears to be less cumbersome and simpler to use than Karpel's. In addition, some theoretical clarifications (such as high-frequency behavior, causality, and stability) have been discussed. Numerical results have been presented that validate the approach proposed.

Certain aspects of the formulation require additional work. For instance, the problem of the instability of the finite-state aerodynamic model that arises when the dimension of the number of additional state variables is increased should be investigated further, to examine whether increasing the number of the sampled aerodynamic data improves the stability. An additional issue that deserves further attention is a comparison of the results obtained with the scheme in comment 6 of the Appendix with those obtained with a direct minimum problem with a reduced number of degrees of freedom. Finally, modeling of nonlinear effects (in particular, the viscous effects, so important in the aerodynamic modeling of control surfaces) should be addresssed.

Appendix: Comments

In this Appendix we present a series of comments to the formulation presented in the main body of this paper.

Comment 1

The least-square procedure used here consists of ensuring that Eq. (11) be satisfied. In the applications, this equation is discretized as

$$\sum_{l,m,n} w(k_n)\ |Z_{lm}(k_n)|^2 = \min \tag{A1}$$

where k_n are uniformely spaced [alternatively, we set $w(k_n) \equiv 1$ and instead use a more dense distribution of k_n in the region where we want a higher weight, see Numerical Results section].

Note that the minimization parameters are the elements of the matrices $\boldsymbol{N}_i$ and $\boldsymbol{D}_i$ and that consequently each element of $\boldsymbol{Z}$ may be minimized independently of the others, i.e., Eq. (A1) is equivalent to

$$\sum_{n} w(k_n)\ |Z_{lm}(k_n)|^2 = \min \tag{A2}$$

for every pair (l, m). Incidentally, this allows for use of a different weight for each pair (l, m).

Next, note that Z_{lm} depend linearly on the minimization parameters. Thus Eq. (A2) represent a minimization problem of the type $(\boldsymbol{By} - \boldsymbol{b})^T \boldsymbol{W}(\boldsymbol{By} - \boldsymbol{b}) = \min$, where $\boldsymbol{W}$ is the diagonal matrix of the weights, and $\boldsymbol{y}$ is the vector of the minimization parameters (i.e., the elements of the matrices $\boldsymbol{N}_i$ and $\boldsymbol{D}_i$). The solution to the problem is given by

$$\boldsymbol{B}^T \boldsymbol{W} \boldsymbol{B} \boldsymbol{y} = \boldsymbol{B}^T \boldsymbol{W} \boldsymbol{b} \tag{A3}$$

Note that the procedure used in Ref. 10 consists in setting $Z_{lm}(k_n) = 0$ at a number of frequencies k_n which is larger than the number of minimization parameters, and then solve the resulting overdetermined set of equations in a least square sense (with suitable weights); of course, the resulting equation is also given by Eq. (A3).

Comment 2

Here, we address the fact that the leading term in Eqs. (17) and (18) is of order p^2. This fact is apparent from a mathematical point of view in the case of potential incompressible flows. Indeed for high frequencies, the first p arises from the boundary condition (relationship between the normal wash and the Lagrangian coordinates), whereas the second is due to the Bernoulli theorem (relationship between pressure and velocity potential). In the integral equation (relationship between velocity potential and normal wash) the contribution of the wake tends to a limit value as the frequency goes to infinity [e.g., $C(k) = \frac{1}{2}$ for $k \to \infty$ in the Theodorsen function, see Ref. 12]; hence, in this case as p tends to infinity the relationship between potential and normal wash becomes independent of p.

One could argue that at high frequency the model of incompressible potential flow does not represent the physics of the phenomenon since compressibility and heat transfer can play an important role. However, we would emphasize that here we are interested in the high-frequency behavior of the mathematical system (potential-incompressible flow) and not in the high-frequency behavior of the physical system.

For the case of linearized (i.e., subsonic and supersonic), unsteady, potential, compressible flows, one applies similar considerations, although a rigorous analysis of the behavior of the integral equation appears desirable; for the case of unsteady viscous flows (linearized around a steady-state configuration) the issue is wide open.

Finally, there have been some questions as to whether the presence of the terms with p^n ($n = 0, 1, 2$) indicates a noncasual behavior of our model. This issue is addressed in comment 7.

Comment 3

It may be worth noting that Eq. (10) with $\boldsymbol{r}(0) = 0$ is fully equivalent to

$$\frac{U_\infty^2}{\ell^2}\ddot{\boldsymbol{q}} + \Omega^2\boldsymbol{q} = q_D\bigg[\boldsymbol{E}_2\ddot{\boldsymbol{q}} + \boldsymbol{E}_1\dot{\boldsymbol{q}} + \boldsymbol{E}_0\boldsymbol{q} + \int_0^t e^{-\boldsymbol{G}(t-\tau_1)}\boldsymbol{F}\dot{\boldsymbol{q}}(\tau_1)\,\mathrm{d}\tau_1\bigg] \tag{A4}$$

since this equation may also be obtained by combining Eqs. (1), (3), and (9) and transforming directly into the time domain. However, the form expressed by Eq. (10) (state-space format) is more convenient in its applications than that of Eq. (A4); in addition, Eq. (10) emphasizes that $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and $\boldsymbol{r}$ determine the "state" of the system (in that if $\boldsymbol{q}, \dot{\boldsymbol{q}}$, and $\boldsymbol{r}$ are known at the time t_0, their knowledge for $t < t_0$ does not add relevant information).

Comment 4

Next, consider some issues related to analytic continuation. In fact, the basis for all of the rational-approximation techniques is the fact that if two analytic functions coincide over the imaginary axis, they will coincide over the whole domain of analyticity (indeed, this is true even if they coincide only on a set of points having an accumulation point[31]). Therefore, a discussion of the nature of the singularities appears appropriate. This nature may be obtained from the fact that the decay of the impulse and indicial responses of any aerodynamic transfer function computed on the basis of potential linear(ized) flow is not an exponential function in time but the inverse of an appropriate power of time (this was pointed out in Ref. 32 and later in Ref. 4; see also Ref. 12, in particular the discussion of the approximation by Sears[33] as compared to that of Jones.[1]) It may be shown that this implies a singularity $r^n \ell n(r)$, n being either 1 for an airfoil or 2 for a lifting surface.

Even if this point is of some theoretical significance, it is scarcely relevant in practical applications, since the original data make it difficult to obtain a correct fit of the singularity mentioned earlier. Moreover, the exact fit of the singularity term prevents a finite-state rational approximation. Indeed, as already mentioned, the available data may possess no singularity, and so its imposition is just an illusory satisfaction of the theory. Finally, it should be noted that from the formulation from Ref. 27 one may see that this singularity is removed if the wake is truncated (the truncation of the wake introduces a difference in the response only after a certain time which increases with the wake length).

Comment 5

Note that if $f(z) = u(z) + iv(z)$ is a strictly proper rational function with all of the poles located on the negative half-plane (and none on the imaginary axis) then we have the relationship

$$u(y) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{v(y')}{y - y'}\,\mathrm{d}y'$$

and

$$v(y) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{u(y')}{y - y'}\,\mathrm{d}y' \tag{A5}$$

where $y := \Im m(z)$. Vice versa, if $f(z)$ is a strictly proper rational function which satisfies Eqs. (A5), then the poles are all on the negative half-plane, with none on the imaginary axis. These relationships are known in physics as dispersion relations[34] and in electrical engineering as the Bayard theorem[35]; they are closely related to the Cauchy integral formula for analytic function[36] and to the Hilbert transform.[37] These equations may be used either to verify stability or, as in Ref. 10, to enrich the data points (while enforcing stability) for a case in which the original aerodynamic matrix is available at too few reduced frequencies (this approach is illustrated in Fig. 6).

Comment 6

Next, consider some issues related to stability and minimum-state reduction of higher order models. Indeed, one could argue that by approximating $\boldsymbol{E}(k)$ with $\hat{\boldsymbol{E}}(k)$ one may introduce a transfer function with poles on the right-hand side of the complex plane; in fact, for this reason, Karpel[6] uses a diagonal matrix for $\bar{\boldsymbol{G}}$ [see Eq. (5)] and enforces the elements to be negative. In our experience, the matrix fraction approach is not affected by this problem if the order of the model is low (e.g., $M = 1$ or 2). This is reasonable in view of what was said in comment 4.

However, as mentioned in the main text one approach considered in Ref. 10 is to use a higher order model in conjunction with standard techniques of reduction to a minimum-state model (e.g., singular-value-based balanced realization and Hankel norm-based reduction) available in widely available software packages.[38,39] This may cause problems, because the use of a higher order model may introduce unstable roots; this problem is removed by truncating the latter[10] and reoptimizing the fit of the reduced model before applying minimum-state reduction techniques.

Indeed, from the point of view adopted in Ref. 10, the matrix fraction approximation i.e., Eqs. (17–21) is used just as a starting point for a minumum-state model which requires, when started with a rough approximation, a costly and slowly converging nonlinear best fit to obtain the matrices of Eq. (19). To this end, Eq. (19) is rewritten as

$$\hat{\boldsymbol{E}}(p) = p^2\boldsymbol{E}_2 + p\boldsymbol{E}_1 + \boldsymbol{E}_0 + \check{\boldsymbol{C}}(p\boldsymbol{I} - \check{\boldsymbol{A}})^{-1}\check{\boldsymbol{B}} \tag{A6}$$

where

$$\check{\mathbf{A}} := \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \qquad \check{\mathbf{B}} := \mathbf{T}^{-1}\mathbf{B} \qquad \check{\mathbf{C}} := \mathbf{C}\mathbf{T} \tag{A7}$$

with $\boldsymbol{T}$ being the matrix of the eigenvectors of $\boldsymbol{A}$. It is noted that $\boldsymbol{T}$ is made up by appending the columns of the eigenvectors corresponding to real eigenvalues and the real and imaginary parts of those corresponding to complex conjugate eigenvalues. Thus, correspondingly $\check{\boldsymbol{A}}$ is a block diagonal matrix with 1×1 blocks for real eigenvalues and 2×2 antisymmetric blocks for complex eigenvalues. The form of Eq. (A6) is clearly maintained after the truncation of the states corresponding to the unstable eigenvalues. The truncation may cause a partial loss of precision in the fit so that a further optimal fit is applied.

Next, it should be noted that the procedure presented in Ref. 10 is slightly different from that used in Ref. 9 and discussed in the main text. Indeed, for the approximation of the generalized unsteady aerodynamic forces to be exact for quasisteady aerodynamics, Ref. 10 enforces an exact fit of the real part and of the slope of the imaginary part of the aerodynamic tranfer function at zero reduced frequency. To satisfy these constraints, the following conditions are imposed:

$$\boldsymbol{E}_0 = \hat{\boldsymbol{E}}(0) + \check{\boldsymbol{C}}\check{\boldsymbol{A}}^{-1}\check{\boldsymbol{B}} \qquad \boldsymbol{E}_1 = \hat{\boldsymbol{E}}'(0) + \check{\boldsymbol{C}}\check{\boldsymbol{A}}^{-2}\check{\boldsymbol{B}} \tag{A8}$$

where the prime denotes the derivative with respect to p, and so Eq. (A6) is rewritten

$$\hat{\boldsymbol{E}}(p) - \hat{\boldsymbol{E}}(0) - p\hat{\boldsymbol{E}}'(0) = p^2\boldsymbol{E}_2 + \check{\boldsymbol{C}}[(p\boldsymbol{I} - \check{\boldsymbol{A}})^{-1} + \check{\boldsymbol{A}}^{-1} + p\check{\boldsymbol{A}}^{-2}]\check{\boldsymbol{B}} \tag{A9}$$

The determination of the remaining $\boldsymbol{E}_2$, $\check{\boldsymbol{C}}$, $\check{\boldsymbol{A}}$, and $\check{\boldsymbol{B}}$, that leads to a best fit is obtained as follows:

1) Calculate $\boldsymbol{E}_2$ and $\check{\boldsymbol{B}}$ by solving the overdetermined linear system of equations obtained by imposing the satisfaction of Eq. (A9) for all of the assigned reduced frequencies for given $\check{\boldsymbol{C}}$ and $\check{\boldsymbol{A}}$.

2) Calculate E_2 and $\check{C}$ by solving the overdetermined linear system of equations obtained by imposing the satisfaction of Eq. (A9) for all of the assigned reduced frequencies for given $\check{B}$ and $\check{A}$.

3) Repeat steps 1 and 2 till no change is produced in E_2, $\check{C}$, and $\check{B}$.

4) Keep E_2, $\check{C}$, and $\check{B}$ fixed and find $\check{A}$ to minimize the sum of the maximum singular values of the fit error matrix over the given reduced frequencies, under the constraint that $\check{A}$ remains stable.

5) Repeat steps 1-4 till convergence

6) At this point a very good and stable fit of the aerodynamic transfer function is obtained which is reduced to minimum states by the minimum-state reduction techniques previously mentioned.

It is then generally advisable to reapply the iteration procedure given by steps 1–5 to the minimum-states model thus obtained to further improve the model.

Comment 7

Here, we consider some questions related to the issue of causality (see comment 2), a concept usually identified with the fact that in a rational transfer function the degree of the numerator is less than that of the denominator [indeed, in Eqs. (17) and (18) the first three terms of the right hand side are sometimes referred to as the noncausal portion of the aerodynamic operator]. To clarify this issue, consider for simplicity a single-input, single-output system. Let u be the input, y be the output, and h be the transfer function of a linear system, such that in the frequency domain

$$\tilde{y} = \tilde{h}\,\tilde{u} \tag{A10}$$

or in the time domain $y = h * u$, where $*$ denotes convolution. The system is called casual if the future does not affect the past. If h is a regular function (and not a generalized function or distribution, see the following), then this requirement implies that $h \equiv 0$ for $t < 0$ (Ref. 26). The use of Jordan's lemma[40] and the inverse Laplace transform guarantees that this condition is satisfied if $\tilde{h}(s)$ goes to zero as s goes to infinity. In particular, if $\tilde{h}(s)$ is a rational function, then this condition requires that the degree of the numerator be smaller than of the denominator. Transfer functions that satisfy this condition are called strictly proper[26] or sometimes causal. We believe that the latter terminology is misleading. For instance, an RLC circuit would be noncausal for $L = 0$. Indeed, in general, if the degree of the numerator is not smaller than that of denominator, then

$$\tilde{h}(s) = \frac{\tilde{n}(s)}{\tilde{d}(s)} = \tilde{q}(s) + \frac{\tilde{r}(s)}{\tilde{d}(s)} \tag{A11}$$

where $\tilde{q}(s)$ is a polynomial in s and $\tilde{r}(s)/\tilde{d}(s)$ is a strictly proper rational function. Note that the contribution $\tilde{q}\tilde{u}$ corresponds in the time domain to a linear combination of the derivatives of u. Therefore, transfer functions that are not strictly proper are not necessarily noncausal, in the sense stated earlier. Incidentally, the inverse Laplace transform of $\tilde{q}$ corresponds to a linear combination of distributions or generalized function (Dirac delta function and its derivatives).

Similar concepts are valid for multi-input/multi-output systems.[26] Indeed, the system $\dot{x} = Ax + B_0 u + B_1 \dot{u}$ (the transfer matrix of which is not strictly proper) may be transformed into the standard form by augmenting the state vector by introducing $x_+ = u$ and the new equation $\dot{x}_+ = v$, where v is the new control vector.

Comment 8

Finally, we show that the finite-state aerodynamic model proposed in Ref. 9 is equivalent to that by Karpel[6] in the case in which N_1 in the Karpel's model is a multiple of N. Specifically, in the case in which $N_1 = NM$, the potential accuracy of the two models is identical, even if Karpel's model has N_1 more constants to be determined than that proposed in Ref. 9.

For simplicity, we consider only the case $M = 1$, i.e., $N_1 = N$ in the Karpel model (for the extension of the equivalence with the Karpel model for the case $M > 1$ see Ref. 9). In this case, the aeroelastic model corresponding to Karpel's equations [see Eq. (5)] is given by

$$\frac{U_\infty^2}{\ell^2}\ddot{q} + \Omega^2 q = q_D(E_2\ddot{q} + E_1\dot{q} + E_0 q + \bar{D}v) \tag{A12}$$

$$\dot{v} + \bar{G}v = \bar{F}\dot{q} \tag{A13}$$

which may be rewritten as Eq. (10) with $r := \bar{D}v$, $G = \bar{D}^{-1}\bar{G}\bar{D}$, and $F := \bar{D}^{-1}\bar{F}$, indicating the full equivalence of the two models. Therefore, the number of the independent unknowns in the least-square process of Ref. 9 [matrices G and F, in Eq. (9)] is only apparently lower than that of the Karpel[6] model [matrices $\bar{D}$, $\bar{F}$, and $\bar{G}$, in Eq. (5)].

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